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DISCRETE PATH INTEGRAL APPROACH TO THE TRACE FORMULA FOR REGULAR GRAPHS

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Abstract. We give a new proof of the trace formula for regular graphs. Our approach is inspired by path integral approach in quantum mechanics, and calculations are mostly combinatorial.

1. Introduction

The famous Selberg's trace formula first appeared in [1]. On a compact hyperbolic surface it relates the eigenvalue spectrum of Laplace operator to the length spectrum of closed geodesics. A version of this formula for finite regular graphs was obtained by Ahumada [2] (cf. also Ihara [3]).

Trace formulae are known to have many implications. For instance, they can be considered as nonabelian generalizations of the Poisson summation formula. In case of finite graphs, since one can find the eigenvalue spectrum of Laplacian for a given graph explicitly, the trace formula lets one find the numbers of closed geodesics of any length (see (34)). In physics trace formulae indicate the cases when semi-classical evaluation of the path integral for state sum of a quantum free particle in some background is exact. Selberg's formula is also known to bear much resemblance to Riemann-Weil formula in number theory.

The original proof of the trace formula for regular graphs (30) was in the framework of "discrete harmonic analysis". We propose another way to derive it, inspired by the path integral approach in quantum mechanics [5]. We consider the trace $Z_\Delta(t) = \mathrm{tr}\, e^{t\Delta}$ as a state sum of the quantum free particle living on the graph. We rewrite it as a sum over closed paths, which is a discrete version of the usual path integral over loops for quantum mechanical state sum. Then we divide the set of closed paths into classes of homotopically equivalent paths. There is a class of contractible paths, and one homotopy class for each closed geodesic on the graph. We explicitly calculate the contribution of each homotopy class to the state sum, thus rewriting it as a contribution of contractible paths plus sum over "long" geodesics (of nonzero length) of contributions of their individual homotopy classes. This is analogous to the stationary phase calculation of the path integral. Geodesics serve as stationary points of the action in the space of loops. Homotopy class of a geodesic serves as a neighbourhood of the stationary point. Thus we arrive to the known trace formula for regular graph, with a specific, physically relevant, choice of test function for eigenvalue spectrum of Laplacian.

We wish to thank P. Zograf who inspired this work.

2. Notations and definitions

Let Γ be a finite regular connected non-oriented graph with vertices of valence $q+1 \geq 2$ with no multiple edges and no edges connecting a vertex with itself. Denote by $V(\Gamma)$ and $E(\Gamma)$ the set of vertices of Γ and the set of edges respectively. Let $|\Gamma| = \#V(\Gamma)$ be the number of vertices. Further denote the space of complex-valued functions on vertices by $\mathrm{Fun}(\Gamma) = \mathbb{C}^{V(\Gamma)}$. A basis function (vector) $|v>$ associated with vertex v

equals 1 on v and 0 on the other vertices. We further adopt the quantum-mechanical notations and denote the transposed basis vector by $< v| = |v >^T$. We call the set of vertices connected to v by edges its link and denote it $\mathrm{Lk}(v)$.

The averaging operator $T : \mathrm{Fun}(\Gamma) \to \mathrm{Fun}(\Gamma)$ acts as follows: for $f \in \mathrm{Fun}(\Gamma)$

$$(Tf)(v) = \sum_{v' \in \mathrm{Lk}(v)} f(v') \tag{1}$$

The Laplace operator Δ on Γ is defined by

$$(\Delta f)(v) = \sum_{v' \in \mathrm{Lk}(v)} f(v') - \mathrm{val}(v) f(v) \tag{2}$$

where $\mathrm{val}(v)$ is the valence of v. Since we consider a regular graph Γ, Δ differs from T by a multiple of identity: $\Delta = -(q+1)\mathbf{1} + T$ where $\mathbf{1}$ is the identity map $\mathrm{Fun}(T) \to \mathrm{Fun}(T)$. The matrix of the averaging operator is just the adjacency matrix of the graph: $< v'|T|v >= 1$ if v and v' are connected by an edge and 0 otherwise. The diagonal elements of T are zero.

The physically interesting quantity is the trace of the heat kernel (the state sum) $e^{t\Delta}$

$$Z_\Delta(t) = \mathrm{tr}\ \exp(t\Delta) = e^{-(q+1)t}\ Z_T(t) \tag{3}$$

where

$$Z_T(t) = \mathrm{tr}\ \exp(tT) \tag{4}$$

It turns out that $Z_T(t)$ is more convenient for our calculation than $Z_\Delta(t)$. If $\{\lambda_j\}_{j=1}^{|\Gamma|}$ is the set of eigenvalues of T then

$$Z_T(t) = \sum_{j=1}^{|\Gamma|} e^{\lambda_j t} \tag{5}$$

If we introduce the spectral density function for T

$$\rho(s) = \sum_{j=1}^{|\Gamma|} \delta(s - \lambda_j) \tag{6}$$

we may express $Z_T(t)$ as a Laplace transform of $\rho(s)$:

$$Z_T(t) = \int_{-\infty}^{\infty} \rho(s) e^{st} ds \tag{7}$$

Eigenvalues λ_j are known to satisfy $-q-1 \le \lambda_j \le q+1$. Moreover, $q+1$ is always an eigenvalue, while $-q-1$ may be an eigenvalue and may be not. If it is, then the distribution of eigenvalues is necessarily even $\rho(-s) = \rho(s)$.

3. Sum over paths

Let us define a closed path of length l as a sequence of vertices $(v_1, \dots, v_l)$ such that for every $j = 1, \dots, l$ the v_j is connected to v_{j+1} by an edge (we identify v_{l+1} with v_1). We will usually omit the word "closed" in the following, since all paths, walks, trajectories etc. will be supposed to be closed. Denote the set of paths by P and the length of a path $p \in P$ by $|p|$. We also denote the number of closed paths of length l by $\mathbf{p}_l$. It is convenient to identify paths of length 0 with vertices of Γ.

Lemma 1.

$$Z_T(t) = \sum_{p \in P} \frac{t^{|p|}}{|p|!} = \sum_{l=0}^{\infty} \mathbf{p}_l \, \frac{t^l}{l!} \tag{8}$$

This expression may be viewed as a discrete version of path integral over loops for the state sum, with $\frac{t^{|p|}}{|p|!}$ being analogue of the measure e^{-S} on loops. We give two different explanations of (8). The first one is more lengthy, but done in the spirit of usual derivation of path integral representation in quantum mechanics. The second is absolutely straightforward and evident.

3.1. **First proof of Lemma 1.** Let us evaluate $Z_T(t)$ in the following manner:

$$Z_T(t) = \sum_{v \in V(\Gamma)} < v|e^{tT}|v >= \lim_{N\to\infty} \sum_{v \in V(\Gamma)} < v|(1+\frac{t}{N}T)^N|v >= \tag{9}$$

$$= \lim_{N\to\infty} \sum_{v_1,\dots,v_N \in V(\Gamma)} < v_1|(1+\frac{t}{N}T)|v_N >< v_N|(1+\frac{t}{N}T)|v_{N-1} > \cdots < v_2|(1+\frac{t}{N}T)|v_1 >$$

we are summing here over all sequences of N vertices $v_1, \dots, v_N$. Notice that matrix elements $< v_{i+1}|(1+\frac{t}{N}T)|v_i >$ equal 1 if $v_{i+1} = v_i$; $\frac{t}{N}$ if v_{i+1} and v_i are connected by an edge; and 0 otherwise. Let us call a walk of length N a sequence of vertices $(v_1, \dots, v_N)$ such that each pair of successive vertices v_j, v_{j+1} are either connected by an edge or coincide. Let W_N be the set of walks of length N. For a walk $w \in W_N$ denote the number of values of j for which v_j and v_{j+1} are connected by an edge by $|w|$.

The only nonzero terms in the last line of (9) are those with the sequence $w = (v_1, \dots, v_N)$ being a walk. For these terms the summand is $(t/N)^{|w|}$. Thus we have

$$Z_T(t) = \lim_{N\to\infty} \sum_{w \in W_N} (t/N)^{|w|} \tag{10}$$

This is also a sort of discrete path integral representation for Z_T. To transform it to the form (8), we need a projection $\pi_N : W_N \to P$ which leaves only those vertices in a walk for which $v_j \neq v_{j+1}$, and forgets the others. For a walk $w \in W_N$ the result of projection $\pi_N(w)$ is a path of length $|w|$. Each path of length l has C_N^l walks as preimages under π_N (C_N^l is a binomial coefficient). So

$$Z_T(t) = \lim_{N\to\infty} \sum_{p \in P} C_N^{|p|} \, (t/N)^{|p|} \tag{11}$$

Using

$$\lim_{N\to\infty} \frac{C_N^{|p|}}{N^{|p|}} = \frac{1}{|p|!} \tag{12}$$

we obtain (8). $\square$

3.2. **Second proof of Lemma 1.** One can arrive to (8) in a more straightforward way: we may just expand the exponent in definition of $Z_T(t)$ in a Taylor series in variable t:

$$Z_T(t) = \mathrm{tr}\; e^{tT} = \sum_{l=0}^{\infty} \frac{t^l}{l!} \,\mathrm{tr}\; T^l \tag{13}$$

then

$$\text{tr } T^l = \sum_{v_1,\ldots,v_l \in V(\Gamma)} < v_1|V|v_l >< v_l|V|v_{l-1} > \cdots < v_2|V|v_1 > \tag{14}$$

the terms in this sum with $(v_1, \ldots, v_l) \in P$ equal 1, all the others vanish; hence

$$\text{tr } T^l = \mathbf{p}_l \tag{15}$$

and we obtain (8). $\square$

4. Sum over geodesics

We use the term “closed trajectory” for equivalence class of closed paths under cyclic permutations of vertices along the path. So a trajectory is a path with information on the starting point forgotten. An elementary homotopy is a transformation of trajectories of the following kind:

$$(v_1, \ldots, v_j, \ldots, v_n) \mapsto (v_1, \ldots, v_j, v', v_j, \ldots, v_n)$$

where $v' \in \text{Lk}(v_j)$. Two trajectories are called homotopic if they can be connected by a chain of elementary homotopies (with arrows either forward or backward). The shortest representative in a homotopy class is called a geodesic trajectory (or just geodesic). An alternative definition of geodesic trajectory is as a trajectory satisfying $v_i \neq v_{i+2}$ for all i. Denote the set of all geodesics on Γ by G. Two paths are called homotopic if their trajectories are homotopic. If $\gamma = (v_1, \ldots, v_n)$ is a trajectory of length n then its r-th power is defined as a trajectory of length rn obtained as γ walked around r times: $\gamma^r = (v_1, \ldots, v_n,\ \ldots,\ v_1, \ldots, v_n)$. A trajectory γ is called primitive if it is not a (non-unit) power of any trajectory. A geodesic trajectory with one of its vertices chosen as a starting point is a geodesic path. A path homotopic to path of length 0 is called contractible. We call geodesics of length 0 short or trivial, and geodesics of length > 0 long.

We proceed now to the calculation of contribution of contractible paths to (8) (one may also call it the contribution of short geodesics).

4.1. **Contribution of contractible paths.** Let us denote $\bar{\Gamma}$ the covering tree for Γ and call some point $C \in \bar{\Gamma}$ the center. The function dist on vertices of the covering tree $\text{dist} : \bar{\Gamma} \to \mathbb{N}_0$ returns the minimal number of edges one must pass to reach given vertex form the center. Any contractible closed path on Γ can be lifted to a closed path on $\bar{\Gamma}$ (and all closed paths there are contractible, since $\bar{\Gamma}$ is a tree) and we adjust the lift so that it start and ends in C. Since each edge passed in one direction by a closed path on $\bar{\Gamma}$ must by passed in the opposite direction, the length of the path must be even. Denote by $P_{2k}(\bar{\Gamma})$ the set of closed paths on $\bar{\Gamma}$ of length $2k$ starting and ending in C; we also need a subset $\tilde{P}_{2k}(\bar{\Gamma}) \subset P_{2k}(\bar{\Gamma})$ consisting of closed paths not returning to C (except the starting point and the end point).

Recall a concept of Dyck path of length $2k$ (see e.g. [4]): it is a sequence of integers $(\alpha_1, \alpha_2, \ldots, \alpha_{2k+1})$ with $\alpha_1 = \alpha_{2k+1} = 0$, $\alpha_i \geq 0$ and $\alpha_{i+1} = \alpha_i \pm 1$. We denote the set of Dyck paths of length $2k$ as D_{2k}; $\#D_{2k} = \text{Cat}_k$ (the k-th Catalan number). There is a projection $\pi_k^D : \tilde{P}_{2k}(\bar{\Gamma}) \to D_{2k-2}$. It acts as follows:

$$(v_1 = C, v_2, \ldots, v_{2k}, v_{2k+1} = C) \mapsto (\text{dist}(v_2) - 1, \ldots, \text{dist}(v_{2k}) - 1) \tag{16}$$

The number of preimages for any Dyck path under π_k^D equals $(q+1)q^{k-1}$ since there are $q+1$ choices to make the step from $v_1 = C$ to v_2; q choices for each step, increasing

dist; steps, decreasing dist are done uniquely (since for any vertex $v \neq C$ of Γ one edge from it leads inward, while the q others lead outward). So we have obtained that

$$\#\tilde{P}_{2k}(\bar{\Gamma}) = (q+1)q^{k-1}\mathrm{Cat}_{k-1} \tag{17}$$

The generating function for the numbers of paths $\#\tilde{P}_{2k}(\bar{\Gamma})$ is obtained as a simple modification of the usual generating function for Catalan numbers:

$$\mathcal{F}_{\tilde{P}}(s) = \sum_{k=1}^{\infty} \#\tilde{P}_{2k}(\bar{\Gamma})\ s^{2k} = (1+q^{-1})\frac{1-\sqrt{1-4qs^2}}{2} \tag{18}$$

For numbers of paths that may pass through the center we obtain

$$\mathcal{F}_P(s) = \sum_{k=0}^{\infty} \#P_{2k}(\bar{\Gamma})\ s^{2k} = \frac{1}{1-\mathcal{F}_{\tilde{P}}(s)} = \frac{(q+1)\sqrt{1-4qs^2}-q+1}{2\ (1-(q+1)^2s^2)} \tag{19}$$

And hence we obtain the contribution to $Z_T(t)$ from contractible paths:

$$[Z_T(t)]_{con} = |\Gamma| \cdot \sum_{k=0}^{\infty} \frac{\#P_{2k}(\bar{\Gamma})}{(2k)!} t^{2k} = |\Gamma| \cdot \frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} ds\ e^{st}\ \frac{1}{s}\mathcal{F}_P(\frac{1}{s}) \tag{20}$$

where real part of A is greater than real parts of all singular points of integrand, as usual for inverse Laplace transform. The factor of $|\Gamma|$ in front is due to the fact that a contractible path can start from any vertex of Γ (we remind that $|\Gamma|$ denotes the number of vertices in Γ). Further evaluating the integral we wrap the contour of integration around the cut $s \in [-2\sqrt{q}, 2\sqrt{q}]$. Thus we proved

Lemma 2. *The contribution of contractible paths to (8) is*

$$[Z_T(t)]_{con} = |\Gamma| \cdot \frac{q+1}{2\pi}\int_{-2\sqrt{q}}^{2\sqrt{q}} ds\ e^{st}\ \frac{\sqrt{4q-s^2}}{(q+1)^2-s^2} \tag{21}$$

In other words, we obtained a contribution to the spectral density of T on Γ from contractible paths:

$$[\rho(s)]_{con} = |\Gamma| \cdot \frac{q+1}{2\pi}\frac{\sqrt{4q-s^2}}{(q+1)^2-s^2} \tag{22}$$

in the interval $s \in [-2\sqrt{q}, 2\sqrt{q}]$ and 0 outside it.

4.2. **Contribution of long geodesics.** Suppose we have a (long) primitive geodesic γ of length l on Γ. To calculate the contribution of its homotopy class to $Z_T(t)$ we need to find the number of paths of length k homotopic to γ: $\mathbf{p}_{\gamma,k}$. Since a path is a trajectory with some point on it chosen as a start/end point, $\mathbf{p}_{\gamma,k} = k\ \mathbf{t}_{\gamma,k}$ where $\mathbf{t}_{\gamma,k}$ is the number of trajectories of length k homotopic to γ. Note that this is not true for non-primitive γ.

We may find the numbers $\mathbf{t}_{\gamma,k}$ using the following combinatorial construction. If $\gamma = (v_1, \ldots, v_l)$ (with the periodic condition $v_0 = v_l$), any trajectory that can be contracted to γ can be represented as a closed contractible path from v_1 to v_1 never going along the edge $v_1 - v_0$; then a step $v_1 - v_2$; then a closed contractible path from v_2 to v_2, never returning along the edge $v_2 - v_1$; then step $v_2 - v_3$ and so on. All in all it is l closed contractible paths with one direction prohibited and l unit steps. Thus we find the generating function for $\mathbf{t}_{\gamma,k}$:

$$\mathcal{F}_{\gamma}^{(traj.)}(s) = \sum_{k=|\gamma|}^{\infty} \mathbf{t}_{\gamma,k} s^k = s^{|\gamma|}(\hat{\mathcal{F}}_P(s))^{|\gamma|} \tag{23}$$

where the superscript $(traj.)$ indicates that we are counting trajectories, $\hat{\mathcal{F}}_P(s)$ is the generating function for the numbers of contractible closed paths with one direction prohibited:

$$\hat{\mathcal{F}}_P(s) = \frac{1}{1 - \frac{q}{q+1}\mathcal{F}_{\tilde{P}}(s)} = \frac{1-\sqrt{1-4qs^2}}{2qs^2} \tag{24}$$

and hence

$$\mathcal{F}_\gamma^{(traj.)}(s) = \left(\frac{1-\sqrt{1-4qs^2}}{2qs}\right)^l \tag{25}$$

For the numbers of paths homotopic to γ we have

$$\mathcal{F}_\gamma^{(paths)}(s) = \sum_{k=|\gamma|}^{\infty} \mathbf{p}_{\gamma,k} s^k = s\frac{\partial}{\partial s}\mathcal{F}_\gamma^{(traj.)}(s) = \frac{|\gamma|}{\sqrt{1-4qs^2}}\left(\frac{1-\sqrt{1-4qs^2}}{2qs}\right)^{|\gamma|} \tag{26}$$

Now we would like to calculate the numbers of paths homotopic to non-primitive geodesic $\gamma = (\gamma')^r$, that is a primitive geodesic γ' passed $r \geq 2$ times. It turns out that if we carry out the scheme above in this case, every path becomes calculated r times. For any geodesic γ denote $\Lambda(\gamma)$ the length of the primitive geodesic γ is power of. If γ is primitive itself, we set $\Lambda(\gamma) = |\gamma|$. Thus for any geodesic we have

$$\mathcal{F}_\gamma^{(paths)}(s) = \frac{\Lambda(\gamma)}{\sqrt{1-4qs^2}}\left(\frac{1-\sqrt{1-4qs^2}}{2qs}\right)^{|\gamma|} \tag{27}$$

Now we have all the information to write down the contribution of a long geodesic γ to $Z_T(t)$:

$$\begin{aligned}[Z_T(t)]_\gamma &= \frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} ds\, e^{st}\, \frac{1}{s}F_\gamma^{(paths)}(\frac{1}{s}) = \\ &= \frac{1}{2\pi i}\int_{A-i\infty}^{A+i\infty} ds\, e^{st}\, \frac{\Lambda(\gamma)}{\sqrt{s^2-4q}}\left(\frac{s-\sqrt{s^2-4q}}{2q}\right)^{|\gamma|}\end{aligned} \tag{28}$$

The last integral reduces to the modified Bessel's function of the first kind $I_{|\gamma|}$. So we deduced

Lemma 3. *For every long geodesic $\gamma \in G$ the contribution of its homotopic class in P to (8) equals*

$$[Z_T(t)]_\gamma = \Lambda(\gamma) q^{-|\gamma|/2} I_{|\gamma|}(2\sqrt{q}t) \tag{29}$$

Collecting together (21) and (29) we obtain the full trace formula:

Theorem (Trace formula for regular graphs). *Let Γ be a finite connected regular graph of valence $q+1 \geq 2$ with $|\Gamma|$ vertices, without multiple edges and edges connecting a vertex to itself; let T be the averaging operator on Γ and $Z_T(t) = \mathrm{tr}\, e^{tT}$ with t a complex variable; let G be the set of long closed geodesics on Γ; for each $\gamma \in G$, $|\gamma|$ is the length of γ and $\Lambda(\gamma)$ is the length of the underlying primitive geodesic: for $\gamma = (\gamma')^r$ with primitive γ' we set $\Lambda(\gamma) = |\gamma'|$. Then*

$$Z_T(t) = |\Gamma| \cdot \frac{q+1}{2\pi}\int_{-2\sqrt{q}}^{2\sqrt{q}} ds\, e^{st}\, \frac{\sqrt{4q-s^2}}{(q+1)^2-s^2} + \sum_{\gamma \in G} \Lambda(\gamma) q^{-|\gamma|/2} I_{|\gamma|}(2\sqrt{q}t) \tag{30}$$

another useful form of the same result is

$$Z_T(t) = |\Gamma| \cdot \frac{q+1}{2\pi} \int_{-2\sqrt{q}}^{2\sqrt{q}} ds\, e^{st} \, \frac{\sqrt{4q-s^2}}{(q+1)^2-s^2} + \sum_{l=3}^{\infty} \mathbf{gp}_l \; q^{-l/2} I_l(2\sqrt{q}t) \tag{31}$$

where $\mathbf{gp}_l$ is the number of geodesic paths of length l. To pass from (30) to (31) one must notice that the number of geodesic paths corresponding to a given geodesic trajectory γ is $\Lambda(\gamma)$.

We may interpret (30) in terms of spectral density $\rho(s)$

Corollary 1.

$$\rho(s) = |\Gamma| \cdot \frac{q+1}{2\pi} \, \frac{\sqrt{4q-s^2}}{(q+1)^2-s^2} \cdot \theta(4q-s^2) + \sum_{\gamma \in G} \Lambda(\gamma) q^{-|\gamma|/2} \frac{T_{|\gamma|}(\frac{s}{2\sqrt{q}})}{\pi\sqrt{4q-s^2}} \cdot \theta(4q-s^2) \tag{32}$$

where

$$T_l(x) = \cos(l \, \arccos(x)) = \frac{1}{2}\left((x+\sqrt{x^2-1})^l + (x-\sqrt{x^2-1})^l\right) \tag{33}$$

is the Chebyshev polynomial of the first kind of degree l.

The sum on the right of (32) is to be understood in the generalized function sense (it does not exist in the ordinary sense since numbers $\mathbf{gp}_l$ grow too fast). Notation θ is used for unit step function. A remarkable fact is that although each geodesic gives a smooth contribution to $\rho(s)$ with support on the interval $s \in [-2\sqrt{q}, 2\sqrt{q}]$ (with singularities on the endpoints), the sum of all contributions is a generalized function with support on eigenvalues of T, scattered across a wider interval $s \in [-q-1, q+1]$.

We may invert (32) in a sense to reproduce the numbers of geodesic paths $\mathbf{gp}_l$ from the spectrum of averaging operator:

Corollary 2. *For $l \geq 1$*

$$\mathbf{gp}_l = 2q^{l/2} \sum_{j=1}^{|\Gamma|} T_l\left(\frac{\lambda_j}{2\sqrt{q}}\right) + \frac{1+(-1)^l}{2}\,(q-1)\,|\Gamma| \tag{34}$$

where λ_j are eigenvalues of the averaging operator T.

In particular since we know the highest eigenvalue $\lambda_{\mathrm{max}} = q+1$, we immediately get (for $q \geq 2$) from (34) the asymptotic law for numbers of geodesic paths: $\mathbf{gp}_l \sim q^l$ as $l \to \infty$ if $-q-1$ is not an eigenvalue. If $-q-1$ belongs to the spectrum of T, the asymptotic is $\mathbf{gp}_l \sim (1+(-1)^l)\; q^l$. For comparison the numbers of all paths behave like $\mathbf{p}_l \sim (q+1)^l$.

4.3. **Case $q = 1$.** This is a simple example where we can check (30) explicitly. The graph Γ is necessarily a polygon with $L \geq 3$ angles. Each long geodesic γ is characterized by the winding number $r \geq 1$ and its direction of movement around the polygon: either clockwise or counterclockwise; $\Lambda(\gamma) = L$ for all γ. The spectrum of T can be found easily: $\lambda_j = 2\cos\frac{2\pi j}{L}$ and the trace formula (30) gives

$$Z_T(t) = \sum_{j=1}^{L} e^{2t\cos\frac{2\pi j}{L}} = L I_0(2t) + 2L\sum_{r=1}^{\infty} I_{rL}(2t) = L \sum_{r=-\infty}^{\infty} I_{rL}(2t) \tag{35}$$

This identity can checked by Poisson resummation. In the limit $L \to \infty$, $t = L^2\tau$ (keeping τ fixed) we recover a special case of the modular transformation for Jacobi

theta function:

$$\sum_{j=-\infty}^{\infty} e^{-4\pi^2 j^2 \tau} = \frac{1}{\sqrt{4\pi\tau}} \sum_{r=-\infty}^{\infty} e^{-\frac{r^2}{4\tau}} \tag{36}$$

4.4. **Remark.** The original trace formula for regular graph in [2] (rewritten in our notations and for special case of trivial character of the fundamental group of Γ) states that for any function $g : \mathbb{Z} \to \mathbb{C}$ such that $g(n) = -g(n)$ for all $n \in \mathbb{Z}$ and

$$\sum_{n=1}^{\infty} |g(n)|\, q^{n/2} < \infty \tag{37}$$

the following holds:

$$\sum_{j=1}^{|\Gamma|} \hat{g}(z_j) = |\Gamma| \cdot \frac{q}{2\pi i} \oint_{|z|=1} \hat{g}(z)\, \frac{1-z^2}{q-z^2}\, \frac{dz}{z} + \sum_{\gamma \in G} \Lambda(\gamma) q^{-|\gamma|/2} g(|\gamma|) \tag{38}$$

where $\hat{g}(z) = \sum_{n=-\infty}^{\infty} g(n) z^{-n}$ and numbers z_j are defined by eigenvalues λ_j by equation $\lambda = \sqrt{q}\, (z + z^{-1})$.

Formula (30) follows from (38) if we choose $g(n) = I_n(2\sqrt{q}t)$ with corresponding $\hat{g}(z) = e^{t\sqrt{q}(z+z^{-1})}$. Integrals representing the contribution of contractible paths convert into one another with the change of variables $s = \sqrt{q}\, (z + z^{-1})$.

4.5. **Remark.** The trace formula (30) is actually valid for graphs with multiple edges and loops. The condition of absence of multiple edges and loops was chosen to simplify the combinatorial constructions.

References

[1] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, Journal of the Indian Mathematical Society 20 (1956) 47-82

[2] Guido Ahumada, Fonctions periodiques et formule des traces de Selberg sur les arbres, C. R. Acad. Sci. Paris 305 (1987), 709-712

[3] Y. Ihara, On discrete subgroup of the two by two projective linear group over p-adic field, J. Math. Soc. Japan 18 3 (1966)

[4] Richard P. Stanley, Enumerative Combinatorics, vol. 1, Wadsworth and Brooks, Monterey, 1986.

[5] Feynman, R. P., and Hibbs, A. R., Quantum Physics and Path Integrals, New York: McGraw-Hill, 1965

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